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Noncompact Leaves of Foliations of Morse Forms

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ABSTRACT. In this paper foliations determined by Morse forms on compact manifolds are considered. An inequality involving the number of connected components of the set formed by noncompact leaves, the number of homologically independent compact leaves, and the number of singular points of the corresponding Morse form is obtained.

KEY WORDS: Morse forms, noncompact leaves of foliations, two-dimensional manifolds.

Consider a Morse form ω that determines a foliation with singularities $\mathcal{F}_\omega$ on a compact manifold M^n. For compact foliations, the author obtained (see [1]) an estimate for the number of homologically independent compact leaves in terms of the number of singular points of the corresponding Morse form. Here we extend this inequality to noncompact foliations.

Further, we obtain an estimate for the number s of connected components of the set formed by noncompact leaves in terms of some characteristics of the Morse form. In [2] Arnoux and Levitt obtained the following estimate of s in terms of characteristics of the manifold M:

$$s \leq \frac{1}{2}\beta_1(M^n). \tag{1}$$

In the present paper we show that these two estimates coincide for two-dimensional manifolds, and that they are independent in the general case.

§1. Basic definitions

Consider a smooth compact connected oriented manifold M of dimension n with a closed 1-form Ω having only Morse singularities; below these forms will be called *Morse forms*. By $\operatorname{Sing}\omega$ denote the set of the singular points of ω. A closed form ω determines a foliation $\mathcal{F}$ of codimension 1 on the set $M \setminus \operatorname{Sing}\omega$.

Let us denote a foliation with singularities $\mathcal{F}_\omega$ on M as follows.

Suppose that the foliation $\mathcal{F}$ is locally determined in the neighborhood of a singular point p, $p \in \operatorname{Sing}\omega$, by the level surfaces of functions f_p such that $f_p(p) = 0$. It is clear that $f_p^{-1}(0)\setminus p \subset \bigcup \gamma_i$, where $\gamma_i \in \mathcal{F}$.

A leaf $\gamma \in \mathcal{F}$ is called a *nonsingular leaf* of $\mathcal{F}_\omega$ if $\gamma \cap f_p^{-1}(0) = \varnothing$ for any $p \in \operatorname{Sing}\omega$. Put

$$F_p = p \cup \{\gamma \in \mathcal{F} \mid \gamma \cap f_p^{-1}(0) \neq \varnothing\} \qquad \text{and} \qquad F = \bigcup_{p \in \operatorname{Sing}\omega} F_p.$$

A *singular leaf* γ_0 of $\mathcal{F}_\omega$ is a connected component of F. It is clear that the number of singular points is finite for Morse forms.

Let us assign the homology class $[\gamma]$ to each nonsingular compact leaf $\gamma \in \mathcal{F}_\omega$. Then the image of the set of nonsingular compact leaves generates a subgroup in $H_{n-1}(M)$. We denote this subgroup by H_ω.

Let p be a singular point of a Morse form ω, and let $x^1, \dots, x^n$ be coordinates in the neighborhood of p such that

$$\omega = \sum_{i=1}^{\lambda} x^i\,dx^i - \sum_{i=\lambda+1}^{n} x^i\,dx^i.$$

Then the *index* $\operatorname{ind} p$ of the singular point p is the number $\min\{\lambda, n-\lambda\}$. By Ω_i denote the set of singular points of index i.

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